



The p -ranks of residual and derived skew Hadamard designs

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ABSTRACT

Let \mathcal{H} be a Hadamard $(4n - 1, 2n - 1, n - 1)$ -design. Suppose that the prime p divides n , but that p^2 does not divide n . A result of Klemm implies that every residual design of \mathcal{H} has p -rank at least n . Also, every derived design of \mathcal{H} has p -rank at least n if $p \neq 2$. We show that when \mathcal{H} is a skew Hadamard design, the p -ranks of the residual and derived designs are at least n even if p^2 divides n or $p = 2$. We construct infinitely many examples where the p -rank is exactly n .

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1. Introduction

The p -rank of an integer matrix N , denoted by $\text{rank}_p(N)$, is the rank of N over the field \mathbb{Z}_p of integers modulo the prime p . The p -rank of a design \mathcal{D} , denoted by $\text{rank}_p(\mathcal{D})$, is the rank of its incidence matrix N . Thus

$$\text{rank}_p(\mathcal{D}) = \text{rank}_p(N).$$

The p -rank is sometimes used to distinguish among non-isomorphic designs, as it is clear that isomorphic designs have equal p -ranks. Moreover, the p -rank is the dimension of the linear code generated by \mathcal{D} over \mathbb{Z}_p .

Let \mathcal{H} be a Hadamard 2-design of order n with parameters $(v, k, \lambda) = (4n - 1, 2n - 1, n - 1)$. The incidence matrix N of \mathcal{H} satisfies

$$NN^T = nI + (n - 1)J.$$

The Hadamard design \mathcal{H} is skew Hadamard provided the points and blocks can be labeled so that its incidence matrix is a tournament matrix of size $4n - 1$, that is, a $(0, 1)$ -matrix satisfying

$$N + N^T = J - I.$$

One must assume that p divides n to glean any useful information about a Hadamard design from its p -rank. The following proposition summarizes some known results [1,5,7] about the p -ranks of Hadamard designs.

Proposition 1. Let \mathcal{H} be a Hadamard $(4n - 1, 2n - 1, n - 1)$ -design. Suppose that the prime p divides n . Then

$$\text{rank}_p(\mathcal{H}) \leq 2n.$$

(a) If p^2 does not divide n , then $\text{rank}_p(\mathcal{H}) = 2n$.

(b) If \mathcal{H} is a skew Hadamard design, then $\text{rank}_p(\mathcal{H}) = 2n$.

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Proposition 1(b) implies that we cannot use p -ranks to distinguish among non-isomorphic skew Hadamard designs. However, we can sometimes use p -ranks to prove that a given Hadamard design is not skew.

When we delete a block and all of its points from a Hadamard design with parameters $(v, k, \lambda) = (4n - 1, 2n - 1, n - 1)$, we obtain a *residual design* with parameters

$$(v, b, r, k, \lambda) = (2n, 4n - 2, 2n - 1, n, n - 1).$$

When we delete a block and all points *not* in the block from a Hadamard design, we obtain a *derived design* with parameters

$$(v, b, r, k, \lambda) = (2n - 1, 4n - 2, 2n - 2, n - 1, n - 2).$$

Label the points and blocks of a Hadamard design so that the incidence matrix N has the form

$$N = \left[\begin{array}{c|c} & \begin{matrix} 1 \\ \vdots \\ 1 \end{matrix} \\ \hline \begin{matrix} N_1 \end{matrix} & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \end{array} \right], \quad (1)$$

where the last column represents the deleted block. The submatrices N_0 and N_1 are the incidence matrices of a residual and a derived design, respectively.

Klemm [6] proved that the p -rank of any balanced incomplete block design with v points is at least $v/2$ provided certain divisibility conditions hold. When applied to residual and derived Hadamard designs, Klemm's result takes the following form.

Proposition 2. Let \mathcal{H} be a Hadamard $(4n - 1, 2n - 1, n - 1)$ -design. If the prime p divides n , but p^2 does not divide n , then

- (a) every residual design of \mathcal{H} has p -rank at least n ;
- (b) every derived design of \mathcal{H} has p -rank at least n if $p \neq 2$.

We will show that when \mathcal{H} is a skew Hadamard design, the p -ranks of the residual and derived designs are at least n even if p^2 divides n or $p = 2$. Here is our main theorem.

Theorem 1. Let \mathcal{H} be a skew Hadamard $(4n - 1, 2n - 1, n - 1)$ -design. Suppose that the prime p divides n . Then the following inequalities hold.

- (a) Every residual design of \mathcal{H} has p -rank at least n .
- (b) Every derived design of \mathcal{H} has p -rank at least n .
- (c) If n is odd and $p \equiv 3 \pmod{4}$, then the p -ranks in (a) and (b) are at least $n + 1$.

We prove **Theorem 1** in Section 3. In Section 4 we construct infinitely many residual and derived skew Hadamard designs of order n with p -rank equal to n . In Section 5 we compute and discuss the p -ranks of all residual and derived skew Hadamard designs of order at most 4.

2. Ranks of tournament matrices

The proof of our main theorem relies on work of de Caen [2,3] and Michael [8] on the p -rank of a tournament matrix. (Also see [4].)

Proposition 3. Let M be a tournament matrix of size m and let p be a prime. Then $\text{rank}_p(M) \geq (m - 1)/2$. Equality cannot hold if $m \equiv 3 \pmod{4}$.

A tournament matrix is *regular* provided every row (and hence every column) has the same number of 1's.

Lemma 1. Let M be a regular tournament matrix of size $2n - 1$. Suppose that the prime p divides n . Then

$$\text{rank}_p(M) \geq n. \quad (2)$$

The inequality is strict if n is odd and $p \equiv 3 \pmod{4}$.

Proof. Observe that M has $n - 1$ 1's in each row and column and that $n - 1 \equiv -1 \pmod{p}$. Thus the columns of M sum to a vector of all -1 's modulo p . Therefore we can append a column of 1's and then a row of 0's to M without changing the p -rank. The rows of the resulting tournament matrix of size $2n$ sum to a vector of all -1 's modulo p . Therefore we can append a row of 1's and a column of 0's to obtain a tournament matrix

$$\widehat{M} = \left[\begin{array}{ccc|cc} & & & 1 & 0 \\ & M & & \vdots & \vdots \\ & & & 1 & 0 \\ \hline 0 & \cdots & 0 & 0 & 0 \\ \hline 1 & \cdots & 1 & 1 & 0 \end{array} \right]$$

of size $2n + 1$ with the same p -rank as M . By [Proposition 3](#)

$$\text{rank}_p(M) = \text{rank}_p(\widehat{M}) \geq \frac{(2n + 1) - 1}{2} = n,$$

and inequality (2) holds.

Suppose that n is odd and $p \equiv 3 \pmod{4}$. Then the tournament matrix \widehat{M} has size $2n + 1$, which is 3 modulo 4, and $\text{rank}(M) = \text{rank}_p(\widehat{M}) \geq n + 1$ by [Proposition 3](#). \square

3. Proof of Theorem 1

Without loss of generality the incidence matrix N of the skew Hadamard design \mathcal{H} is a tournament matrix of size $4n - 1$ of the form

$$N = \left[\begin{array}{ccc|c} & & & 1 \\ & N_1 & & \vdots \\ & & & 1 \\ \hline & & & 0 \\ & N_0 & & \vdots \\ & & & 0 \end{array} \right] = \left[\begin{array}{cc|c} M_1 & * & \begin{matrix} 1 \\ \vdots \\ 1 \end{matrix} \\ \hline * & M_0 & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \\ \hline 0 & \cdots & 0 & 1 & \cdots & 1 & 0 \end{array} \right].$$

The matrices M_1 and M_0 are regular tournament matrices of size $2n - 1$. For both $i = 1$ and $i = 0$ [Lemma 1](#) gives

$$\text{rank}_p(N_i) \geq \text{rank}_p(M_i) \geq n.$$

Also, if n is odd and $p \equiv 3 \pmod{4}$, then strict inequality holds by [Lemma 1](#). \square

4. A construction

We now construct an infinite class of examples for which equality holds in [Theorem 1](#). Suppose that n is even and let \widetilde{N} be the incidence matrix of a skew Hadamard design of order $n/2$ with parameters

$$(v, k, \lambda) = (2n - 1, n - 1, (n/2) - 1).$$

Then $\text{family rank}_p(\widetilde{N}) = n$ by [Proposition 1\(b\)](#). The matrix

$$N = \left[\begin{array}{cc|c} \widetilde{N} & J - \widetilde{N} & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \\ \hline \widetilde{N} & \widetilde{N} & \begin{matrix} 1 \\ \vdots \\ 1 \end{matrix} \\ \hline 1 & \cdots & 1 & 0 & \cdots & 0 & 0 \end{array} \right] \tag{3}$$

is readily verified to be the incidence matrix of a skew Hadamard design of order n . Let p be a prime that divides $n/2$. Then the p -rank of the derived Hadamard design using the last block equals

$$\text{rank}_p([N \mid \widetilde{N}]) = \text{rank}_p(\widetilde{N}) = n.$$

Also, because \tilde{N} is a regular tournament matrix of size $2n - 1$, the last row of N is the sum of the first $2n - 1$ rows. Therefore the p -rank of the residual Hadamard design using the last block equals

$$\text{rank}_p([\tilde{N} \mid J - \tilde{N}]) = \text{rank}_p([\tilde{N} \mid J]) = \text{rank}_p(\tilde{N}) = n.$$

Thus equality holds in parts (a) and (b) of [Theorem 1](#).

Skew Hadamard designs are known to exist for infinitely many orders (see [9], e.g.) and are conjectured to exist for all orders. Thus our construction gives infinitely many instances of equality in our main theorem for residual and derived designs of even order.

5. Examples

We used Maple to compute the p -ranks of all residual and derived skew Hadamard designs of orders $n = 2, 3$, and 4 .

Example. ($n = 2$): The unique Hadamard $(7, 3, 1)$ -design is skew. Every residual and derived design has 2-rank equal to 3. Strict inequality holds in [Theorem 1](#)(a) and (b).

Example. ($n = 3$): The unique Hadamard $(11, 5, 2)$ -design is skew. Every residual and derived design has 3-rank equal to 5. Strict inequality holds in [Theorem 1](#)(a)–(c).

Example. ($n = 4$): There are five Hadamard $(15, 7, 3)$ -designs. The three non-skew designs have 2-ranks equal to 5, 6, and 7. The two skew designs have 2-rank equal to 8 in accordance with [Proposition 1](#)(b).

- (i) One skew Hadamard design is of the form (3), where \tilde{N} is an incidence matrix of a $(7, 3, 1)$ -design. Every residual (derived) design has 2-rank equal to 6 (resp., 5) except for a unique residual (derived) design with 2-rank equal to 4.
- (ii) The incidence matrix of the other skew Hadamard design is the transpose of the incidence matrix in (i). There are seven residual (derived) designs with 2-rank equal to 5 and eight residual (derived) designs with 2-rank equal to 7.

Note that the two skew Hadamard $(15, 7, 3)$ -designs can be distinguished from one another by the parity of the 2-rank of any residual design; a residual design whose 2-rank is even (odd) must arise from the skew Hadamard design in (i) (resp., (ii)).

Equality holds in [Theorem 1](#)(a) and (b) for the residual and derived designs in (i) with 2-rank equal to 4.

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